MEROMORPHIC MAPS OF KÄHLER MANIFOLDS WITH TRIVIAL CANONICAL BUNDLES

DO DUC THAI AND DUC-VIET VU

arXiv:1611.03373v1 [math.CV] 10 Nov 2016

Abstract. Let M be a (bounded or not) domain of $\mathbb{C}^n$ which is complete with respect to a Kähler metric, or more generally, a complete Kähler manifold with trivial canonical bundle. Let f be a linearly nondegenerate meromorphic map from M to the complex projective space $\mathbb{P}^m$. Under an assumption on the positivity of the pull-back by f of the Fubini-Study form on $\mathbb{P}^m$, we prove that f can not omit a certain number of hyperplanes in subgeneral position in $\mathbb{P}^m$. This is deduced directly from a non-integrated defect relation for such f which generalizes that obtained by Fujimoto in the case where M is a ball.

1. Introduction

Let (M, ω_M) be a complete Kähler manifold with trivial canonical bundle. Typical examples for such M are open subsets of $\mathbb{C}^n$ endowed with complete Kähler metrics. Let $f : M \to \mathbb{P}^m$ be a meromorphic map. Let N be a positive integer. Let $\{H_j\}_{1\le j\le q}$ be a family of hyperplanes in N-subgeneral position in $\mathbb{P}^m$, *i.e,* for every $1 \le j_1 < \cdots < j_{N+1} \le q$ we have $\bigcap_{k=1}^{N+1} H_{j_k} = \varnothing$. Recall that $\mathrm{d^c} = i(\bar\partial - \partial)/(4\pi)$, hence $\mathrm{dd^c} = i\partial\bar\partial/(2\pi)$. Let

$$\omega := \mathrm{dd^c} \log(|w_1|^2 + \cdots + |w_{m+1}|^2)$$

be the Fubini-Study form of $\mathbb{P}^m$, where $[w_1 : \cdots : w_{m+1}]$ are the homogeneous coordinates on $\mathbb{P}^m$. Let $\mathrm{Ric}\,\omega_M$ be the Ricci form of ω_M given by

$$\mathrm{Ric}\,\omega_M := -i\partial\bar\partial \log\det[\omega_{M;j,k}],$$

where locally $\omega_M = i\sum_{1\le j\le k\le n} \omega_{M;j,k} dz_j \wedge d\bar z_k$ with $n = \dim M$. Recall that $1/(2\pi)\mathrm{Ric}\,\omega_M$ is contained in the first Chern class $c_1(M)$ of M. The following is our first main result.

Theorem 1.1. *Let M, ω_M, f, ω be as above. Assume that f is linearly nondegenerate, and there exist a positive constant $\rho \ge 0$ and a bounded measurable function h on M such*

2000 *Mathematics Subject Classification.* Primary 32H30; Secondary 32H04, 32H25, 14J70.

Key words and phrases. N-subgeneral position, defect relation, Ricci curvature.

The research of the authors is supported by an NAFOSTED grant of Vietnam (Grant No. 101.04-2014.48).

that $\log|h|^2$ *is locally integrable and*

$$\rho f^*\omega + dd^c \log|h|^2 + \frac{\mathrm{Ric}\,\omega_M}{2\pi} \geq 0. \tag{1}$$

Then f *can not omit* $(2N-m+1)+\rho m(2N-m+1)$ *hyperplanes in* N*-subgeneral position in* $\mathbb{P}^m$.

Since the canonical line bundle of every (bounded or not) domain of $\mathbb{C}^n$ is trivial, we deduce immediately the following.

Corollary 1.2. *Let* M *be a domain of* $\mathbb{C}^n$ *such that* M *is complete with respect to the metric induced by a Kähler form* ω_M *on* M*. Let* $f : M \to \mathbb{P}^m$ *be a linearly nondegenerate meromorphic map. Assume that (1) holds. Then* f *can not omit* $(2N-m+1)+\rho m(2N-m+1)$ *hyperplanes in* N*-subgeneral position in* $\mathbb{P}^m$.

In the case where $\mathrm{Ric}\,\omega_M$ is positive, the assumption (1) is automatically satisfied for $h \equiv 1$ and $\rho = 0$. Observe that given a bounded domain V in $\mathbb{C}^n$ we can choose a family of hyperplanes in general position of $\mathbb{P}^n$ such that the inclusion $i : V \hookrightarrow \mathbb{C}^n \subset \mathbb{P}^n$ does not intersect that family. Hence we can obtain the following byproduct which might be of interest.

Corollary 1.3. *On an arbitrary bounded domain of* $\mathbb{C}^n$*, there exists no Kähler complete metric with positive Ricci curvature.*

Corollary 1.3 can be put in the context of the study of noncompact complete Kähler manifolds with positive curvatures. Actually, under a stronger hypothesis on the positivity of curvature, it is expected that such manifold should be biholomorphic to $\mathbb{C}^n$; see [13]. Theorem 1.1 is a direct consequence of the following result.

Theorem 1.4. *Let the hypothesis be as in Theorem 1.1. Then we have*

$$\sum_{j=1}^{q} \bar{\delta}_f^{[m]}(H_j) \leq (2N-m+1)+\rho m(2N-m+1), \tag{2}$$

for every family $\{H_j\}_{1\leq j\leq q}$ *of hyperplanes in* $\mathbb{P}^m$ *in* N*-subgeneral position.*

Here, for a hyperplane H of $\mathbb{P}^m$, the number $\bar{\delta}_f^k(H)$ is a refined version of the Fujimoto non-integrated defect truncated to level $k \in \mathbb{N}$ of f with respect to a divisor H in $\mathbb{P}^m$, see (16) in Section 3. In some sense, the defect $\bar{\delta}_f^k(H)$ quantifies the intersection $f(M) \cap H$. In a more general situation, we also obtain a similar estimate as in Theorem 1.4 for meromorphic maps from M to a compact manifold with divisor targets; see Theorem 4.3 in Section 4.

When M is a ball in $\mathbb{C}^n$ and $\{H_j\}_{1\le j\le q}$ is in general position, (2) is proved by Fujimoto in [8, Th. 5.10] where he introduced his notion of non-integrated defect. Note that the Ricci form in the last paper is $-\frac{1}{2\pi}$ times that we defined before. The result of Fujimoto has been generalized to several different situations when hyperplanes are replaced by hypersurfaces, see [17, 19, 20, 15]. A common point of these papers is that they always assume that M is a complete Kähler manifold whose universal covering is a ball in $\mathbb{C}^n$. This in fact reduces the problem to the case where M is a ball. Hence, the interesting point in our results is that they hold, in particular, for *any* (bounded or not) domain of $\mathbb{C}^n$ which is complete with respect to a given Kähler form. Using our techniques and those in [17, 19, 20, 15], we can generalize without difficulty the results there to the case where M is complete Kähler manifold whose universal covering has a trivial canonical line bundle.

We now describe the main ideas in proving Theorem 1.4. We will prove Theorem 1.4 by contradiction. The main difficulty in our proof is the lack of a reasonable Nevanlina theory for meromorphic maps from M. So far such a theory has been only available for holomorphic maps from parabolic manifolds; see [5, 18] for more information. Hence, one can not apply directly arguments in [8] to get the defect relation (2). This is the reason why recent results on this direction always required the condition that the universal covering of M is the unit ball in $\mathbb{C}^n$ which would clearly reduce the question to the case of maps from the unit ball.

To deal with the above mentioned problem, a key argument is to use a result of Fornaess and Stout [6] saying that there exist an open subset U of M biholomorphic to the unit polydisc $\mathbb{D}^n$ of $\mathbb{C}^n$ and $M\setminus U$ is of zero Lebesgue measure. We then construct a Nevanlina theory for the restriction f_U of f to $U \approx \mathbb{D}^n$. As in [8], to the data $(f, \{H_j\})$ we will associate a *global nonconstant* psh function w_1 on M. The next step is to bound from above the volume form $e^{w_1}\omega_M^n$ by a measure on $U \approx \mathbb{D}^n$ depending on the characteristic function of f_U. If (2) were wrong, the last measure would be of finite mass on U. We deduce that $\int_U e^{w_1}\omega_M^n$ is finite, hence so is $\int_M e^{w_1}\omega_M^n$ because $M \setminus U$ is of zero measure. This is a contradiction, see Proposition 3.8 below.

Inspired by the Nevanlinna theory for holomorphic mappings into compact complex manifolds (see [3]), in the last section, we establish a generalization of Theorem 1.4 for meremorphic maps to a compact manifold.

The paper is organized as follows. In Section 2, we present the Nevanlina theory for meromorphic maps from polydiscs. In Section 3, we prove Theorem 1.4. In Section 4, we prove Theorem 4.3 which is a generalization of Theorem 1.4 for meremorphic maps to a compact manifold. Finally, we would like to remark that this paper is a corrected

version of our previous preprint [4] whose proofs contain flaws.

Acknowledgements. The second author would like to thank Dinh Tuan Huynh and Duc Quang Si for fruitful discussions. This work was completed during a stay of the first author at the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the staff there, in particular the partial support of VIASM.

2. Second main theorem for meromorphic maps from polydiscs

In this section we present Nevanlina theory for meromorphic maps from polydiscs. The general strategy is the same as in the case of meromorphic maps from balls.

Firstly we fix some notations. For a positive real number $r \in (0,1]$, define $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. Let $\partial'\mathbb{D}^n_r := \{|z_j| = r : 1 \leq j \leq n\}$. We will identify $\partial'\mathbb{D}^n_r$ with $[0,2\pi]^n$ via the isomorphism

$$r(e^{it_1}, \cdots, e^{it_n}) \longmapsto t = (t_1, \cdots, t_n).$$

When $r = 1$, we write $\mathbb{D}$ instead of $\mathbb{D}_1$. Let $\mathbb{P}^m$ be the complex projective space of dimension m and ω the Fubini-Study form there. Denote by $\|\cdot\|$ the canonical Hermitian metric on the hyperplane line bundle of $\mathbb{P}^m$. Let $f = (f_1, \cdots, f_{m+1})$ be a meromorphic map from the polydisc $\mathbb{D}^n$ into $\mathbb{P}^m$, where f_j is holomorphic function on $\mathbb{D}^n$ for $1 \leq j \leq m+1$ and $\cap_{j=1}^{m+1}\{f_j = 0\}$ is of codimension ≥ 2. Define

$$g := \max\nolimits_{1 \leq j \leq n} \log |z_j|$$

which is the pluricomplex Green function on $\mathbb{D}^n$ with pole at the origin, see [12, 1]. Fix a constant $r_0 \in (0,1)$. *The characteristic function of f* is defined by

$$T_f(r) := \int_{\log r_0}^{\log r} ds \int_{\{g<s\}} f^*\omega \wedge (\mathrm{dd^c} g^2)^{n-1}.$$

Let H be a hyperplane of $\mathbb{P}^m$. For $1 \leq k \leq \infty$, *the truncated counting function of f to level k with respect to H* is defined by

$$N_f^{[k]}(r,H) := \int_{\log r_0}^{\log r} ds \int_{\{g<s\}} \min\{[f^*H], k\} \wedge (\mathrm{dd^c} g^2)^{n-1}$$

and *the proximity function* is

$$m_f(r,H) = \frac{1}{(2\pi)^n} \int_{\partial'\mathbb{D}^n_r} \log \frac{1}{||H \circ f||^2}\, dt.$$

For simplicity, we omit the superscript $^{[k]}$ when $k = \infty$. Applying the Lelong-Jensen formula to g (see [1]), we have

$$T_f(r) = N_f(r,H) + m_f(r,H) - m_f(r_0,H) \tag{3}$$

and

(4)
$$T_f(r) = \frac{1}{(2\pi)^n}\int_{\partial'\mathbb{D}^n_r} \log\|f\|^2 dt + O(1), \quad N_f(r,H) = \frac{1}{(2\pi)^n}\int_{\partial'\mathbb{D}^n_r} \log\|H\circ f\|^2 dt + O(1)$$

where $\|f\|^2 := |f_1|^2 + \cdots + |f_{m+1}|^2$.

Recall that $\log^+ x := \max\{\log x, 0\}$ for $x \in \mathbb{R}^+$. For an n-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of non-negative integers, put $|\alpha| := \sum_{j=1}^n \alpha_j$. In what follows, the notation $\lesssim$ means $\leq$ up to a multiplicative constant independent of r.

Proposition 2.1. *Let g be a meromorphic function on $\mathbb{D}^n$. Let $\alpha \in (\mathbb{Z}^+)^n$ and p, p' positive real numbers. Assume that $p|\alpha| < p' < 1$. Then, there exist a constant C independent of r and a subset $E \subset [0,1]$ satisfying $\int_E \frac{dr}{1-r} < \infty$ such that for all $r \in [r_0, 1)\setminus E$ we have*

$$\int_{\partial'\mathbb{D}^n_r} \Big|\frac{D^\alpha g}{g}\Big|^p dt \leq C|1-r|^{p'} T_g(r)^{p'} \tag{5}$$

and

$$m_{\frac{D^\alpha g}{g}}(r,\infty) \leq C\big(\log^+ T_g(r) + \log|1-r|^{-1}\big). \tag{6}$$

Proof. We have

$$m_{\frac{D^\alpha g}{g}}(r,\infty) \leq \int_{\partial'\mathbb{D}^n_r} \log\Big(\Big|\frac{D^\alpha g}{g}\Big| + 1\Big) dt \leq \log \int_{\partial'\mathbb{D}^n_r} \Big|\frac{D^\alpha g}{g}\Big| dt + O(1)$$

by concavity of log function. Using the last inequality, we observe that (6) is deduced directly from (5) by choosing $p = 1/(3|\alpha|), p' = 3p/2$. Hence it remains to prove (5). We will prove it by induction on $|\alpha|$.

Let $r' \in (r_0, 1)$. Consider the case where $|\alpha| = 1$. Without loss of generality, we can suppose $\alpha = (1, 0, \cdots, 0)$. Fix $z' := (z_2, \cdots, z_n)$ such that $\tilde{g}_{z'} := g(\cdot, z')$ is a meromorphic function on $\mathbb{D}$. Note that the last condition holds for almost everywhere $z' \in \partial'\mathbb{D}^{n-1}$. Let $0 < p_1 < p_1' < 1$ be real numbers. Either using [8, Th. 3.1] in dimension one or using directly Riesz's representation formula for $\tilde{g}_{z'}$, one obtains

$$\int_{\partial\mathbb{D}_r} \Big|\frac{Dg}{g}\Big|^{p_1} dt \lesssim |r'-r|^{-p_1'} T^{p_1'}_{\tilde{g}_{z'}}(r'), \tag{7}$$

for every $r \in [r_0, r')$. Integrating (7) on $z' \in \partial'\mathbb{D}^{n-1}_r$ gives

(8)
$$\int_{\partial'\mathbb{D}^n_r} \Big|\frac{Dg}{g}\Big|^{p_1} dt \lesssim C|1-r|^{-p_1'} \int_{z'\in\partial'\mathbb{D}^{n-1}_r} T^{p_1'}_{\tilde{g}_{z'}}(r') \lesssim C|1-r|^{-p_1'} \Big\{\int_{z'\in\partial'\mathbb{D}^{n-1}_r} T_{\tilde{g}_{z'}}(r')\Big\}^{p_1'}.$$

Write $g = g_1/g_2$, where g_1, g_2 are holomorphic on $\mathbb{D}^n$ and have no common divisor. Put $\|g\|^2 := |g_1|^2 + |g_2|^2$. By (4), we have

$$T_{\tilde{g}_{z'}}(r') = \frac{1}{2\pi}\int_{\partial\mathbb{D}_{r'}} \log\|g\|^2 dt + O(1).$$

Integrating the last equality on $z' \in \partial'\mathbb{D}_r^{n-1}$ yields

$$\int_{z'\in\partial'\mathbb{D}_r^{n-1}} T_{\tilde{g}_{z'}}(r') = \frac{1}{2\pi}\int_{\partial\mathbb{D}\times\partial'\mathbb{D}_{r'}^{n-1}} \log\|g\|^2 dt + O(1) \lesssim \int_{\partial'\mathbb{D}_{r'}^n} \log\|g\|^2 dt = T_g(r') \tag{9}$$

because $\log\|g\|^2$ is psh. Combining (9) and (8) gives

$$\int_{\partial'\mathbb{D}_r^n} \left|\frac{Dg}{g}\right|^{p_1} dt \lesssim |1-r|^{-p_1'} T_g^{p_1'}(r'). \tag{10}$$

Choose $r' = r + (1-r)/(eT_{\tilde{g}_{z'}}(r'))$. Using (10) and [10, Le. 2.4], we obtain (5) for $\alpha = (1, 0, \cdots, 0)$.

Now we suppose that (5) holds for $\alpha' \in (\mathbb{Z}^+)^n$ in place of α, where $|\alpha'| < |\alpha|$. As already observed, (6) also holds for α' in place of α. By (3) and (4), we get

$$\begin{aligned} T_{D^{\alpha-1}g}(r) &\le T_{D^{\alpha-1}g/g}(r) + T_g(r) + O(1) \\ &\le m_{D^{\alpha-1}g/g}(r,\infty) + N_{D^{\alpha-1}g/g}(r,\infty) + T_g(r) + O(1) \\ &\le C\big(T_g(r) + \log^+ T_g(r) + \log|1-r|^{-1}\big) \end{aligned} \tag{11}$$

by the induction hypothesis and the fact that $N_{D^{\alpha-1}g/g}(r,\infty) \le N_g(r,\infty) \le T_g(r) + O(1)$. Let $\{\alpha_k\}_{1\le k\le|\alpha|}$ be an increasing sequence of n-tuples satisfying

$$\alpha_1 = 0, \quad |\alpha_k| = |\alpha_{k-1}| + 1,$$

for all $k \ge 2$. Let $p|\alpha| < p'' < p'$. By applying (5) to $(D^{\alpha_{k-1}}, \alpha_k - \alpha_{k-1}, p|\alpha|)$, there is a subset $E \subset [r_0, 1)$ with $\int_E dr/(1-r) < \infty$ such that for $r \in [r_0,1)\backslash E$ and $1 \le k \le |\alpha|$, we have

$$\int_{\partial'\mathbb{D}_r^n} \left|\frac{D^{\alpha_k}g}{D^{\alpha_{k-1}}g}\right|^{p|\alpha|} \lesssim |1-r|^{p''} T_{D^{\alpha_{k-1}}g}(r)^{2p''} \lesssim |1-r|^{p''} T_g(r)^{2p''} + C|1-r|^{p''}. \tag{12}$$

On the other hand, observe that

$$\int_{\partial'\mathbb{D}_r^n} \left|\frac{D^{\alpha}g}{g}\right|^p dt = \int_{\partial'\mathbb{D}_r^n} \prod_{k=2}^{|\alpha|} \left|\frac{D^{\alpha_k}g}{D^{\alpha_{k-1}}g}\right|^p dt \le \frac{1}{|\alpha|}\sum_{k=2}^{|\alpha|} \int_{\partial'\mathbb{D}_r^n} \left|\frac{D^{\alpha_k}g}{D^{\alpha_{k-1}}g}\right|^{p|\alpha|} dt$$

which is

$$\lesssim |1-r|^{p''} \sum_{k=1}^{|\alpha|} T_{D^{\alpha_{k-1}}g}(r)^{p''} + |1-r|^{p''} \lesssim |1-r|^{p'} T_g(r)^{p'}$$

by (12) and (11). The proof is finished. □

Assume that f is linearly nondegenerate. By [8, Pro. 4.10], there exist $\alpha_1, \cdots, \alpha_{m+1} \in \mathbb{N}^n$ such that

$$|\alpha_1|+\cdots+|\alpha_{m+1}|\leq \frac{m(m+1)}{2}, \quad |\alpha_k|\leq m \quad (1\leq k\leq m+1) \tag{13}$$

and the generalized Wronskian of f

$$W_{\alpha_1,\cdots,\alpha_{m+1}}(f) := \det(D^{\alpha_k} f_j : 1 \leq k, j \leq m+1) \not\equiv 0.$$

Moreover, for such $\alpha_1, \cdots, \alpha_{m+1}$, we have

$$\left(\frac{W_{\alpha_1,\cdots,\alpha_{m+1}}(f)}{f_1\cdots f_{m+1}}\right)_\infty \leq \sum_{j=1}^{m+1} \min\{(f_j)_0, m\}. \tag{14}$$

Remark 2.2. By [8], such $\alpha_1, \cdots, \alpha_{m+1}$ also exist for every meromorphic map $f = (f_1, \cdots, f_{m+1})$ from a complex manifold M to $\mathbb{P}^m$, where $f_1, \cdots, f_{m+1}$ are holomorphic functions on M with $\cap_{j=1}^{m+1}\{f_j = 0\}$ is of codimension ≥ 2 in M.

Using Proposition 2.1, (14) and then repeating usual arguments in the proof of the Cartan-Nochka theorem (see [16]), we get

Theorem 2.3. *Let $H_1, \cdots, H_q$ be hyperplanes of $\mathbb{P}^m$ in N-subgeneral position. Let $f : \mathbb{D}^n \to \mathbb{P}^m$ be linearly nondegenrate meromorphic mapping. Then we have*

$$(q-2N+m-1)T_f(r) \leq \sum_{j=1}^{q} N_f^{[m]}(r, D_j) + O\big(\log^+ T_f(r)\big) + O(\log|1-r|^{-1}),$$

for $r \in [r_0, 1]\backslash E$ where $E \subset [0, 1]$ with $\int_E dr/(1-r) < \infty$.

3. Non-integrated defect relation

Let M be a n-dimensional complete Kähler manifold with the Kähler form ω_M. Let f be linearly nondegenerate meromorphic map from M to $\mathbb{P}^m$. Let H be a hyperplane in $\mathbb{P}^m$ and $k \in \mathbb{N}\cup\{\infty\}$. The *defect of f with respect to H truncated to level k* is defined by

$$\delta_f^{[k]}(H) = \liminf_{r\to 1}\left(1 - \frac{N_f^{[k]}(r, H)}{T_f(r)}\right).$$

Let $\mathcal{A}$ be the set of positive real numbers η satisfying the following condition: there exists a bounded function h on M such that $\log|h|^2$ is locally integrable on M and

$$\eta f^*\omega + dd^c \log h^2 \geq \min\{k, f^*H\} \tag{15}$$

on M in the sense of currents. Note that $1 \in \mathcal{A}$ because (15) holds when $\eta = 1$ and $h = \|H\circ f\|$. The *non-integrated defect of f with respect to H truncated to level k* is defined by

$$\bar{\delta}_f^{[k]}(H) := 1 - \inf\{\eta : \eta \in \mathcal{A}\}. \tag{16}$$

When $k=\infty$, we denote $\bar{\delta}_f(H) := \bar{\delta}_f^{[\infty]}(H)$.

Remark 3.1. In Fujimoto's definition of the non-integrated defect (see [8]), it is required that h, h/φ are continuous, where φ is a holomorphic function in M such that $(\varphi)_0 = \min\{k, f^*H\}$. In the above refined version of non-integrated defect, we are able to remove the last constrain thanks to Proposition 3.8 below which holds for *every plurisubharmonic functions* u on M. Thus, the defect $\bar{\delta}_f^{[k]}(H)$ is à priori greater than or equal to the original one of Fujimoto.

By [6, Th. 1], there exists an open subset U of M such that U is biholomorphic to the polydisc $\mathbb{D}^n$ and $M\setminus U$ has a zero measure. *Throughout the paper, we fix a such open subset U and identify it with $\mathbb{D}^n$.* Denote by f_U the restriction of f to U.

Lemma 3.2. *We have the following properties of the non-integrated defect:*

(i) $0 \le \bar{\delta}_f^{[k]}(H) \le 1$,

(ii) $\bar{\delta}_f^{[k]}(H) = 1$ *if* $f(M)\cap H = \varnothing$,

(iii) $\bar{\delta}_f^{[k]}(H) \ge 1 - k/k_0$ *if* $f^*H \ge k_0 \min\{f^*H, 1\}$, *where* $k_0 \in \mathbb{N}^*$,

(iv) *if*

$$\lim_{r\to 1} T_{f_U}(r) = \infty,$$

then $0 \le \bar{\delta}_f^{[k]}(H) \le \delta_{f_U}^{[k]}(H) \le 1$,

(v) *let $\tilde{M}$ be the universal covering of M and $\pi : \tilde{M} \to M$ the canonical projection, then we have $\bar{\delta}_f^{[k]}(H) \le \bar{\delta}_{f\circ\pi}^{[k]}(H)$, for every hyperplane H and $k \in \mathbb{N}$.*

Proof. The first two properties and (v) are obvious. We now prove (iii). Put $\eta := k/k_0, h := \|H\circ f\|^\eta$. We have

$$\eta f^*\omega + \mathrm{dd^c}\log h^2 = \eta f^*H \ge \min\{f^*H, k\}$$

because $f^*H \ge k_0\min\{f^*H,1\}$. Therefore, $\eta \in \mathcal{A}$. This implies (iii).

Let us prove (iv). Let φ be a holomorphic function on $\mathbb{D}^n$ such that $(\varphi)_0 = \min\{f_U^*H, k\}$. Let $\eta\in\mathcal{A}$ and h as in (15). Write $f_U = (f_{U,1},\cdots,f_{U,m+1})$ which is a reduced representation of f_U. Put

$$v := \eta\log(|f_{U,1}|^2+\cdots+f_{U,m+1}|^2) + \log h^2 - \log|\varphi|^2.$$

By choice of η, v is psh on $\mathbb{D}^n$. Combining this with (4), we obtain, for $r\in[r_0,1)$,

$$\begin{aligned}\int_{\partial'\mathbb{D}^n_{r_0}} v dt \le \int_{\partial'\mathbb{D}^n_r} v dt &= (2\pi)^n\big(\eta T_{f_U}(r) - N^{[k]}_{f_U}(r,H)\big) + \int_{\partial'\mathbb{D}^n_r}\log h^2 dt + O(1)\\ &= (2\pi)^n\big(\eta T_{f_U}(r) - N^{[k]}_{f_U}(r,H)\big) + O(1)\end{aligned}$$

because h is bounded on M. As a consequence, we get

$$1-\frac{N^{[k]}_{f_U}(r,H)}{T_{f_U}(r)}\geq 1-\eta+\frac{O(1)}{T_{f_U}(r)}.$$

Letting $r\to 1$ gives (iv). The proof is finished. $\square$

The next part of this section is devoted to the proof of Theorem 1.4. Let the notations be as in the statement of Theorem 1.4. Firstly observe that we can suppose that M is simply connected. The general case follows from that one because by Property (v) of Lemma 3.2, if $\tilde{M}$ is the universal covering of M and $\pi:\tilde{M}\to M$ the canonical projection, we can study $f\circ\pi$ instead of f. Hence from now on suppose that M *is simply connected.* Our strategy follows the original one of Fujimoto. Roughly speaking, the idea is that if (2) were wrong, we would construct a nonconstant psh function u on M with the help of the restriction of f to U such that $\int_M e^u\omega_M^n=\infty$. This contradicts Proposition 3.8 below. Hence we get Theorem 1.4.

We begin with auxiliary lemmas. Let f_U be as above.

Lemma 3.3. *Assume that*

$$\limsup_{r\to 1}\frac{T_{f_U}(r)}{-\log(1-r)}=\infty.$$

Then (2) holds.

Proof. By Theorem 2.3, we obtain $\sum_{j=1}^q\delta_f^{[m]}(H_j)\leq 2N-n+1$. Combining this with Property (iv) of Lemma 3.2 gives (2). The proof is finished. $\square$

Lemma 3.4. *The current* $\mathrm{dd^c}\log\|f\|^2$ *is nonzero.*

Proof. By a direct computation, we get

$$\mathrm{dd^c}\log\|f\|^2=\frac{1}{2\pi}\|f\|^{-4}\sum_{1\leq j<k\leq m+1}|f_jf_k|^2\big(f_j^{-1}\partial f_j-f_k^{-1}\partial f_k\big)\wedge\overline{\big(f_j^{-1}\partial f_j-f_k^{-1}\partial f_k\big)}$$

outside $V:=\cup_{j=1}^{m+1}\{f_j=0\}$. Hence if $\mathrm{dd^c}\log\|f\|^2=0$ then $f_j^{-1}\partial f_j=f_k^{-1}\partial f_k$ for every $1\leq j<k\leq m+1$. On any local simply connected chart V_1 outside V, this implies that $\partial(\log f_j-\log f_k)=0$ there. Since the function $(\log f_j-\log f_k)$ is holomorphic on V_1, the last equality yields $f_j=cf_k$ for some constant c. Hence, f is not linearly nondegenerate, a contradiction. The proof is finished. $\square$

Let I_f be the indeterminacy locus of f which is of codimension at least 2 in M. Hence, the fundamental group of $M\backslash I_f$ is equal to that of M which is trivial; see [9, Th. 2.3]. This implies that $M\backslash I_f$ is also simply connected. The last property allows us to lift f to the covering $(\mathbb{C}^{m+1})^*$ of $\mathbb{P}^m$. We denote also by f that lift. Thus there exist global

holomorphic functions $f_1, \cdots, f_{m+1}$ defined on $M \backslash I_f$ such that $f = (f_1, \ldots, f_{m+1})$. Since $\mathrm{codim} I_f \geq 2$, we can extend f_j holomorphically through I_f.

Let $\alpha_1, \cdots, \alpha_{m+1}$ be as in (14) for f, see Remark 2.2. Let $\{\omega(j)\}_{1 \leq j \leq q}$, $\tilde{\omega}$ be the Nochka weights and the Nochka constant of $\{H_j\}_{1 \leq j \leq q}$, see [16]. Recall that they are nonnegative constants and

$$\sum_{j=1}^{q} \omega(j) = \tilde{\omega}(q - 2N + m - 1) + m + 1 \tag{17}$$

and

$$\frac{m+1}{2N - m + 1} \leq \tilde{\omega} \leq \frac{n}{N} \cdot \tag{18}$$

Define

$$\chi := \frac{|W(f)(z)|}{\prod_{j=1}^{q} \left| H_j\big(f(z)\big) \right|^{\omega(j)}} \cdot$$

We can assume that $\sum_{j=1}^{q} \bar{\delta}_f^{[m]}(H_j) > 2N - m + 1$, since otherwise we get (2). Thus, by definition of the non-integrated defect, there exist $\eta_j \geq 0$ $(1 \leq j \leq q)$ and bounded nonnegative functions h_j such that

$$\eta_j f^* \omega + dd^c \log h_j^2 \geq \min\{m, f^* H_j\} \tag{19}$$

and $s := \sum_{i=1}^{q} (1 - \eta_j) > 2N - m + 1$. Let ϵ be a small positive constant. Let ρ be as in (1). Put

$$\beta := \frac{2(\rho + \epsilon)}{\tilde{\omega}(s - 2N + m - 1)} \cdot$$

Let σ be a *nowhere vanishing holomorphic* section of the canonical line bundle of M which is a holomorphic differential $(n, 0)$-form on M. Hence $\sigma \wedge \bar{\sigma}$ defines a volume form on M and

$$\mathrm{Ric}\, \omega_M = 2\pi dd^c |\omega_M^n / (\sigma \wedge \bar{\sigma})|.$$

Let Φ be a biholomorphism from $\mathbb{D}^n$ to U. The next step in our proof is the following.

Lemma 3.5. *There exists a nonconstant psh function w_1 on M such that*

$$e^{w_1} \omega_M^n \leq \|f\|^{\beta \tilde{\omega}(q - 2N + m - 1)} |\chi|^{\beta} \sigma \wedge \bar{\sigma}. \tag{20}$$

Proof. By (14) and the properties of Nochka weights, we get

$$\sum_{1 \leq j \leq q} \omega(j)(f^* H_j - \min\{f^* H_j, m\}) \leq \big(W(f)\big)_0. \tag{21}$$

Let $K := \max\{\|h_1\|_{L^\infty}^2, \cdots, \|h_{m+1}\|_{L^\infty}^2, \|h\|_{L^\infty}^2\}$. Put

$$u_j := \log \frac{h_j^2}{K} + \log^{\eta_j} \|f\|^2, \quad v := \log \chi + \tilde{\omega} \sum_{j=1}^{q} u_j.$$

By (19) and (21), v is a psh function on M and

$$e^v \le \|f\|^{\tilde{\omega}\sum_{j=1}^q \eta_j}|\chi| = \|f\|^{\tilde{\omega}(q-s)}|\chi|. \tag{22}$$

Let h be as in (1). Set

$$w := \log \frac{\|f\|^{2(\rho+\epsilon)}h^2\sigma\wedge\bar{\sigma}}{K\omega_M^n}, \quad w_1 := w + \beta v.$$

By (1), we have

$$e^w\omega_M^n \le \|f\|^{2\rho+2\epsilon}\sigma\wedge\bar{\sigma} \tag{23}$$

and $\mathrm{dd^c}w \ge \epsilon\mathrm{dd^c}\log\|f\|^2$, where the last form is nonzero on M by Lemma 3.4. Hence, $\mathrm{dd^c}w_1 \ge \epsilon\mathrm{dd^c}\log\|f\|^2$. We deduce that w_1 is a nonconstant psh function on M. Combining (23) and (22) gives (20). The proof is finished. $\square$

Lemma 3.6. *Assume that $\beta m(m+1)/2 < 1$. Let β' be a real number so that $\beta m(m+1)/2 < \beta' < 1$. Then we have*

$$\int_M e^{w_1}\omega_M^n \le C_1\int_0^1 |1-r|^{\beta'}\big[T_{f_U}(r)\big]^{\beta'}dr, \tag{24}$$

for some positive constant C_1.

Proof. The hypothesis and (13) gives

$$\beta\sum_{j=1}^{m+1}|\alpha_j| < 1. \tag{25}$$

Let ω_0 be the canonical Kähler form on $\mathbb{C}^n$. Since Φ is a biholomorphisim, there exist a nowhere vanishing holomorphic function φ on $\mathbb{D}^n$ for which

$$\Phi^*(\sigma\wedge\bar{\sigma}) = |\varphi|^2\omega_0^n \tag{26}$$

on $\mathbb{D}^n$. Since the last open set is simply connected and φ is a nowhere vanishing holomorphic function there, we can find a holomorphic function $\tilde{\varphi}$ on $\mathbb{D}^n$ with $\varphi = e^{\tilde{\varphi}}$; see, for example, [14, p.22]. Put

$$t := \frac{2}{\beta\big(\tilde{\omega}(q-2N+m-1)+1\big)}.$$

Observe that $f_U' := e^{t\tilde{\varphi}}f_U$ is a reduced representation of f_U. Thus $T_{f_U'}(r) = T_{f_U}(r) + O(1)$ as $r\to 1$. Put

$$\chi_U' := \frac{|W(f_U')(z)|}{\prod_{j=1}^q \big|H_j\big(f_U'(z)\big)\big|^{\omega(j)}} = |e^{-\tilde{\omega}(q-2N+m-1)\tilde{\varphi}}|\frac{|W(f_U)(z)|}{\prod_{j=1}^q \big|H_j\big(f_U(z)\big)\big|^{\omega(j)}}$$

by [8, Pro. 4.9] and (17). Taking into account (20), (26) and the last equality, one get

$$\int_M e^{w_1}\omega_M^n = \int_U e^{w_1}\omega_M^n \le \int_{\mathbb{D}^n} \|f_U\|^{\beta\tilde{\omega}(q-2N+m-1)}|\chi\circ\Phi|^\beta \Phi^*(\sigma\wedge\bar{\sigma}) = \int_{\mathbb{D}^n} \|f'_U\|^{\beta\tilde{\omega}(q-2N+m-1)}|\chi'_U|^\beta \omega_0^n. \tag{27}$$

Using (25), Proposition 2.1, then arguing as in [16, Le. 4.2.3] (see also [8, Pro. 6.1]), one obtains

$$\int_{\partial'\mathbb{D}_r^n} \|f'_U\|^{\beta\tilde{\omega}(q-2N+m-1)}|\chi'_U|^\beta dt \le C|1-r|^{\beta'} T_{f'_U}(r)^{\beta'} \le C_1|1-r|^{\beta'} T_{f_U}(r)^{\beta'}, \tag{28}$$

for $r\in[r_0,1]$ outside a set E with $\int_E dr/(1-r)<\infty$, where C, C_1 are constants independent of r. Actually, the set E can be chosen to be the union of a countable number of intervals in $[0,1]$. By increasing C_1 if necessary as in [7, Pro. 5.5], (28) holds for every $r\in[r_0,1]$. Hence, combining (27) and (28), we get (24). The proof is finished. □

Corollary 3.7. *Assume that*

$$\limsup_{r\to 1} \frac{T_{f_U}(r)}{-\log(1-r)} < \infty. \tag{29}$$

Then $\beta m(m+1)/2 \ge 1$. In particular, (2) holds.

Proof. Suppose to the contrary that $\beta m(m+1)/2 < 1$. Let β as in Lemma 3.6. By the last lemma and (29), we get $\int_M e^{w_1}\omega_M^n < \infty$. This contradicts Proposition 3.8. The proof is finished. □

End of the proof of Theorem 1.4. Combining Corollary 3.7 and Lemma 3.3, we obtain

$$\beta \ge \frac{2}{m(m+1)}.$$

This is equivalent to

$$s \le 2N-m+1+\frac{(\rho+\epsilon)m(m+1)}{\tilde{\omega}} \le 2N-m+1+(\rho+\epsilon)m(2N-m+1),$$

for every $\epsilon>0$ (by (18)). Letting $\epsilon\to 0$ gives (2). The proof is finished. □

Proposition 3.8. *Let u be a nonconstant psh function on M. Then*

$$\int_M e^u \, d\,\mathrm{vol} = \infty. \tag{30}$$

Proof. Let p be an integer greater than 2. Fix a point $a\in M$. Let $B(a,r)$ be the ball of radius r centered at a of M (with respect to the metric induced by ω_M). We will prove the following stronger statement: for every non-constant nonnegative psh function v on M we have

$$\liminf_{r\to\infty} \frac{1}{r^2}\int_{B(a,r)} v^p \, d\,vol = \infty. \tag{31}$$

The equality (30) is just a consequence of (31) if we take $v = e^{u/p}$. For simplicity, we denote by B_r the ball $B(a, r)$. Now assume that (31) were wrong. Then, there exists a constant A and a sequence $\{r_j\}$ such that $r_0 = 1$, $r_{j+1} \geq 2r_j$ and

$$\frac{1}{r_j^2}\int_{B_{r_j}} v^p \, d\,vol \leq A,$$

for every j. In order to get a contradiction, we will modify the proof of [11, Th. 2.1]. The last theorem applies to nonnegative continuous subharmonic functions v on M. The hypothesis on the continuity is needed there because they used a theorem of Greene-Wu which says that a such v can be approximated by global $\mathcal{C}^2$ subharmonic functions. In our present situation, we can drop that assumption and use the usual convolution of psh functions to regularize v instead of the one of Greene-Wu.

Precisely, by taking convolution of v with a suitable cut-off function, for every $j \in \mathbb{N}$, there is a decreasing sequence $\{v_{k,j}\}_{k\in\mathbb{N}}$ of C^∞-nonnegative psh functions on $B(r_{j+2})$ such that $v_{k,j}$ converge pointwise to v on $B(r_{j+1})$. By the monotone convergence there is $k_j \in \mathbb{N}$ for which

$$\frac{1}{r_j^2}\int_{B_{r_j}} v_{k_j,j}^p \, d\,vol \leq \frac{1}{r_j^2}\int_{B_{r_j}} v^p \, d\,vol + 1. \tag{32}$$

Define $v_j := v_{k_j,j}$. For each $j \geq 1$, let φ_j be a Lipschitz continuous function such that $0 \leq \varphi_j \leq 1$ and $\varphi_j(x) \equiv 1$ on B_{r_j} and $\varphi_j(x) \equiv 0$ on $M \setminus B_{r_{j+1}}$ and $\mathrm{grad}\varphi_j \leq C/r_j$ a.e on M, where C is a constant which does not depend on j (see [11, Le. 1]). Let ϵ be a positive real number. For real number $j, N \in \mathbb{N}$ with $j \leq N$ put

$$Q_{j+1}^N(\epsilon) := \int_{B_{r_{j+1}}} \varphi_j^2 (v_{N+1}^2 + \epsilon)^{\frac{p-2}{2}} ||\mathrm{grad}\, v_{N+1}||^2 \, d\,vol$$

and $Q_{j+1}^N := \lim_{\epsilon\to 0} Q_{j+1}^N(\epsilon)$. We will prove that $\lim_{N\to\infty} Q_{N+1}^{2N} = 0$. Suppose that this were wrong. Then there exist a sequence $N_k \to \infty$ and a real number $\delta_0 > 0$ for which $Q_{N_k+1}^{2N_k} \geq \delta_0$ for every $k \in \mathbb{N}$. Since $Q_{j+1}^N \geq Q_{j'+1}^N$ for $j \geq j'$, we get

$$Q_{j+1}^{2N_k} \geq Q_{N_k+1}^{2N_k} \geq \delta_0 \tag{33}$$

for $j \geq N_k$. Letting $\epsilon \to 0$ in (37) of Lemma 3.9 below, we have

$$(Q_{j+1}^N)^2 \leq C_p[Q_{j+1}^N - Q_j^N]\int_{B_{r_{j+1}}} |v_{N+1}|^p d\,vol \leq AC_p[Q_{j+1}^N - Q_j^N] \tag{34}$$

by (32). As a consequence, one obtains

$$Q_{j+1}^N \leq AC_p \tag{35}$$

for $j \leq N$. Choose $N = 2N_k$. Taking the sum over $N_k \leq j \leq 2N_k$ of (34) yields

$$\sum_{N_k \leq j \leq 2N_k} (Q_{j+1}^{2N_k})^2 \leq AC_p Q_{2N_k+1}^{2N_k} \leq (AC_p)^2$$

by (35). On the other hand, by (33), the left-hand side of the last inequality is $\geq N_k \delta_0$. This implies $(AC_p)^2 \geq N_k \delta_0$ for every $k \in \mathbb{N}$. Letting $k \to \infty$ in the last inequality gives a contradiction. Hence, we obtain $\lim_{N\to\infty} Q_{N+1}^{2N} = 0$ which implies

$$\lim_{k\to+\infty} \int_{B_{r_{N_k}}} v_{2N_k}^{p-2} \|\operatorname{grad} v_{2N_k}\|^2 = 0. \tag{36}$$

We can choose $p \in \mathbb{N}$ even. Put $q := (p-2)/2$. Let $\varphi \in \mathcal{C}^2(M)$ with compact support in M. We have

$$\int_M v^{q+1} \Delta\varphi = \lim_{k\to+\infty} \int_{B_{r_{N_k}}} v_{N_k}^{q+1} \Delta\varphi = -(q+1) \lim_{k\to+\infty} \int_{B_{r_{N_k}}} v_{N_k}^{q} \langle \operatorname{grad} u_{N_k}, \operatorname{grad}\varphi \rangle = 0$$

by (36). In the other words, $\Delta v^{q+1} = 0$ in the sense of currents. Hence, $v^{q+1} \in C^\infty$ by the regularity theorem. Put $X = \operatorname{grad} v^{q+1}$. We have

$$\begin{aligned}\int_M \|X\|^2 \varphi &= \int_M \langle \operatorname{grad} v^{q+1}, \varphi X\rangle = -\int_M v^{q+1} \operatorname{div}(\varphi X) \\ &= -\lim_{k\to+\infty} \int_{B_{r_{N_k}}} v_{N_k}^{q+1} \operatorname{div}(\varphi X) = (q+1) \lim_{k\to+\infty} \int_{B_{r_{N_k}}} v_{N_k}^{q+1} \langle \operatorname{grad} v_{N_k}, \varphi X\rangle = 0\end{aligned}$$

by (36). Therefore, $X = 0$. That means u is constant, a contradiction. The proof is finished. □

Lemma 3.9. *We have*

$$\left(Q_{j+1}^N(\epsilon)\right)^2 \leq C_p [Q_{j+1}^N(\epsilon) - Q_j^N(\epsilon)] \int_{B_{r_{j+1}}} (v_{N+1}^2 + \epsilon)^{\frac{p-2}{2}} |v_{N+1}|^2 dvol, \tag{37}$$

for $0 \leq j \leq N$ and a positive constant C_p independent of N, j, v_N.

Proof. This is the inequality (2.8) in [11]. □

4. A generalization

In this section, we establish a generalization of Theorem 1.4 for meremorphic maps to a compact manifold. Let $\mathcal{L} \to X$ be a holomorphic line bundle over a compact complex manifold X of dimension n. Let m be a nonnegative integer. Fix a positive integer d. Let E be a vector subspace of $H^0(X, \mathcal{L}^d)$ of dimension $m+1$ and $\{c_k\}_{k=1}^{m+1}$ a basis of E. Denote by $B(E)$ the base locus of E. Define $\Phi : X \setminus B(E) \to \mathbb{P}^m$ by

$$\Phi(x) := [c_1(x) : \cdots : c_{m+1}(x)].$$

Recall that Φ can be extended to be a meromorphic map from X to $\mathbb{P}^m$. Denote by rank$E$ the maximal rank of Jacobian of Φ on $X \setminus B(E)$ which does not depend on the choice of a basis of E.

Let $N \geq n$ and $q \geq N+1$ be integers. Let $d_1, d_2, \ldots, d_q$ be divisors of d. Let σ_j $(1 \leq j \leq q)$ be in $H^0(X, \mathcal{L}^{d_j})$ such that $\sigma_j^{d/d_j} \in E$ for $1 \leq j \leq q$. Denote by D_j the divisor of σ_j. Assume that $\{D_j\}_{1\leq j\leq q}$ *is in N-subgeneral position with respect to E, i.e,* for any $1 \leq i_0 < \cdots < i_N \leq q$, we have $\bigcap_{j=0}^N D_{i_j} = B(E)$. Put

$$u := \mathrm{rank}E, \quad b := \mathrm{dim}B(E) + 1 \text{ if } B(E) \neq \varnothing, \quad b := -1 \text{ otherwise.}$$

For $1 \leq j \leq q$, write $\sigma_j^{d/d_j} = \sum_{1\leq j\leq m+1} a_{jk}c_k$, where $a_{jk} \in \mathbb{C}$. Put $H_j := \sum_{1\leq j\leq m+1} a_{jk}z_k$, where $[z_1, \cdots, z_{m+1}]$ are the homogeneous coordinates of $\mathbb{P}^m$. Put $Q := \{1, 2, \ldots, q\}$ and

$$c(K) := \mathrm{rank}\{H_j\}_{j\in K}, \quad n_0(\{D_j\}) := \max\{c(K) : K \subset Q \text{ with } |K| \leq N+1\} - 1,$$

for each $K \subset Q$. We also set

$$n(\{D_j\}) := \max\{c(K) : K \subset Q\} - 1.$$

Observe that

$$u \leq n_0(\{D_j\}) \leq n(\{D_j\}) \leq m.$$

Set $k_N := 2N - u + 2 + b, s_N := n_0(\{D_j\})$ and $t_N := \dfrac{u-b}{n(\{D_j\}) - u + 2 + b}$.

Proposition 4.1. *(*[3, Pro. 2.11]*) Assume that $u > b$ and $q \geq k_N$. Then, there exist positive constant $\omega(j)$ for $1 \leq j \leq q$ and $\tilde{\omega}$ such that the following three conditions hold:*

(i) $0 < \omega(j) \leq \tilde{\omega} \leq 1\,(1 \leq j \leq q)$ *and* $\tilde{\omega} \geq t_N/k_N$.

(ii) $\sum_{1\leq j\leq q} \omega(j) \geq \tilde{\omega}(q - k_N) + t_N$.

(iii) *for $1 \leq j \leq q$, let E_j be arbitrary positive real numbers and R a subset of Q of cardinality $N+1$. Then, there exist indexes $j_1, \cdots, j_{s_N+1}$ in R such that*

$$\bigcap_{1\leq i\leq s_N+1} H_{j_i} \cap Y = \bigcap_{j\in R} H_j \cap Y$$

and

$$\sum_{j\in R} \omega(j)E_j \leq \sum_{1\leq k\leq s_N+1} E_{j_k}.$$

The numbers $\omega(j), \tilde{\omega}$ are called Nochka weights and Nochka constant for $\{D_j\}_{1\leq j\leq d}$.

Let $f : \mathbb{D}^n \to X$ be an analytically nondegenerate meromorphic map with respect to E, i.e $f(\mathbb{D}^n)$ is not contained in any divisor of E and $\overline{f(\mathbb{D}^n)} \cap B(E) = \varnothing$. Put

$$\omega_E := \mathrm{dd^c} \sum_{1\leq k\leq m+1} |c_k(f)|^2.$$

Fix a constant $r_0 \in (0,1)$, we define

$$T_f(r,E) := \int_{\log r_0}^{\log r} ds \int_{\{g<s\}} f^*\omega_E \wedge (\mathrm{dd^c} g^2)^{n-1},$$

$$N_f^{[k]}(r,D_j) := \frac{d}{d_j}\int_{\log r_0}^{\log r} ds \int_{\{g<s\}} \min\{[f^*D_j],k\} \wedge (\mathrm{dd^c} g^2)^{n-1},$$

for $1 \le j \le q$. Set

$$K_{E,N,\{D_j\}} := \frac{(m+1)k_N(s_N-u+2+b)}{t_N}.$$

By repeating the argument in [3], we get the following second main theorem.

Theorem 4.2. *Let $X, \{D_j\}_{1\le j\le q}, u, b, f$ be as above. Assume that $u > b$. Then, for all $r \le [r_0,1)$ outside a set $E \subset [0,1]$ with $\int_{E_j} \frac{dr}{1-r} < \infty$, we have*

$$\big(q - K_{E,N,\{D_j\}}\big)T_f(r,E) \le \sum_{j=1}^{q} N_f^{[md/d_j]}(r,D_j) + O\big(\log^+ T_f(r)\big) + O(\log|1-r|^{-1}).$$

We remark that in case where X is a complex projective space, a more refined second main theorem is obtained if one use the techniques in [2].

Now, let (M,ω_M) be a complete Kähler manifold whose canonical bundle is trivial. Let $f: M \to X$ be an analytically nondegenerate meromorphic map with respect to E, *i.e*, $f(M)$ is not contained in any divisor of E and $\overline{f(M)} \cap B(E) = \varnothing$. Put

$$\omega_E := \mathrm{dd^c} \sum_{1\le k\le m+1} |c_k(f)|^2.$$

For $0 \le k \le \infty$, denote by $\mathcal{A}_E$ the set of positive real numbers η satisfying the following condition: there exists a bounded function h on M such that $\log|h|^2$ is locally integrable on M and

$$\eta f^*\omega_E + \mathrm{dd^c}\log h^2 \ge \frac{d}{d_j}\min\{k, f^*D_j\} \tag{38}$$

on M in the sense of currents. We define

$$\bar{\delta}_{f,E}^{[k]}(D_j) := 1 - \inf\{\eta : \eta \in \mathcal{A}_E\}.$$

Note that this definition does not depend on choosing a basis of E. The following theorem is a generalization of Theorem 1.4.

Theorem 4.3. *Let $X, \{D_j\}_{1\le j\le q}, u, b, f$ be as above. Assume that $u > b$ and there exist a positive constant $\rho \ge 0$ and a bounded measurable function h on M such that* $\log|h|^2$ *is locally integrable and*

$$\rho f^*\omega_E + \mathrm{dd^c}\log|h|^2 + \frac{\mathrm{Ric}\,\omega_M}{2\pi} \ge 0. \tag{39}$$

Then we have

$$\sum_{j=1}^{q} \bar{\delta}_{f,E}^{[md/d_j]}(D_j) \leq K_{E,N,\{D_j\}} + \frac{\rho k_N m(m+1)}{t_N}. \tag{40}$$

Proof. Since $f(M) \cap B(E) = \emptyset$, $F := \Phi \circ f$ is a well-defined meremorphic map from M to $\mathbb{P}^m$. Moreover, F is linearly nondegenerate because f is nondegenerate with respect to E. It should be noted that $F(M)$ is contained in the subvariety $\Phi(X)$ of dimension u. Let H_j be as above. Clearly, we have $\bar{\delta}_{f,E}^{[kd/d_j]}(D_j) = \bar{\delta}_F^{[k]}(H_j)$ and (1) can be rewritten as

$$\rho F^* \omega + \mathrm{dd^c} \log |h|^2 + \frac{\mathrm{Ric}\, \omega_M}{2\pi} \geq 0. \tag{41}$$

We are now almost being in the same situation as in Theorem 1.4. The difference is that the family $\{H_j\}_{1 \leq j \leq q}$ *is not in N-subgeneral position.* However, we still have Nochka weights $\omega(j)$ and $\tilde{\omega}$ for $\{H_j\}_{1 \leq j \leq q}$ in the sense given in Proposition 4.1. As before, we can assume that M is simply connected and F is given by global holomorphic functions $F_1, \dots, F_{m+1}$ on M. In order to prove (40), compared with the proof of Theorem 1.4, we only need to make some minor modifications which are in fact implicitly introduced in the proof of [3, Th. A]. We just recall here very briefly. As in the proof of [3, Th. A], we can add some hyperplanes $H_{q+1}, \dots, H_{q+m-u+b+1}$ to the family $\{H_j\}$ so that

$$\{H_j\}_{j \in R} \cup \{H_j\}_{q+1 \leq j \leq q+m-u+b+1}$$

is in $(s_N + m - u + b + 1)$-subgeneral position, for every $R \in Q$ of cardinality $s_N + 1$ such that $\bigcap_{j \in R} D_j = B(E)$. Now we will bound the sum of the defects of F with respect to $\{H_j\}_{1 \leq j \leq q+m-u+b+1}$ rather than the original family $\{H_j\}_{1 \leq j \leq q}$. In fact, we are going to prove a stronger statement

$$\sum_{j=1}^{q+m-u+b+1} \bar{\delta}_F^{[m]}(H_j) \leq K_{E,N,\{D_j\}} + \frac{\rho k_N m(m+1)}{t_N}. \tag{42}$$

Define

$$\chi^E := \frac{|W(F)(z)|^{s_N - u + 2 + b}}{\prod_{j=1}^{q+m-u+b+1} \big|H_j\big(F(z)\big)\big|^{\omega(j)}}.$$

By Inequality (12) in [3], the function χ^E plays exactly the same role as χ does in the proof of Theorem 1.4. Now using Inequalities (12) and (14) in [3] and proceeding as in the last proof, we can construct a nonconstant psh function w_1^E on M such that $\int_M e^{w_1^E} \omega_M^n < \infty$ if (42) did not hold. Thus we get the desired result by Proposition 3.8. The proof is finished. $\square$

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Do Duc Thai,

Department of Mathematics, Hanoi National University of Education, 136 XuanThuy str., Hanoi, Vietnam

E-mail address: `doducthai@hnue.edu.vn`

Duc-Viet Vu,
UPMC Univ Paris 06, UMR 7586, Institut de Mathématiques de Jussieu, 4 place Jussieu, F-75005 Paris, France.

E-mail address: duc-viet.vu@imj-prg.fr